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Oscillation Properties of Generalized Characteristic Polynomials for Totally Positive and Positive Definite Matrices

SAMUEL KARLIN*
The Weizmann Institute of Science
Rehovot, Israel,
and Stanford University
Stanford, California

AND

ALLAN PINKUS
The Weizmann Institute of Science
Rehovot, Israel

Recommended by Hans Schneider

1. INTRODUCTION

A key fact necessary to characterize spaces of splines interpolating data at all integers is that the $n-2k+1$ degree polynomial $(0 < k \leqslant n/2)$

$$T_{k,n}(\lambda) = \begin{vmatrix} \binom{k}{0} & \binom{k+1}{0} & & \cdots & & \binom{n}{0} \\ \binom{k}{1} & \binom{k+1}{1} & & \cdots & & \binom{n}{1} \\ \vdots & \vdots & & & & \vdots \\ \binom{k}{k}-\lambda & \binom{k+1}{k} & & \cdots & & \binom{n}{k} \\ \binom{k}{k+1} & \binom{k+1}{k+1}-\lambda & & \cdots & & \binom{n}{k+1} \\ \vdots & \vdots & & & & \vdots \\ \binom{k}{n-k} & \cdots & \binom{n-k-1}{n-k} & \binom{n-k}{n-k}-\lambda & \cdots & \binom{n}{n-k} \end{vmatrix}, \tag{1.1}$$

possesses exactly $n-2k+1$ simple real zeros of sign $(-1)^k$ (see Lipow

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and Schoenberg [10], Schoenberg and Sharma [14]). The elements below the diagonal involving the λ terms are zero as $\binom{l}{k} = 0$ for $l < k$. This result set forth in Lipow and Schoenberg was proved by seemingly ad hoc methods involving special manipulations and devices.

Stimulated to penetrate the essence of the fact of (1.1) we uncovered the easily proved Theorem 1.1 below.

To state the result we fix some notation and terminology. Let $B = \|b_{ij}\|_{i=1}^{n}{}_{j=1}^{m}$ be an $n \times m$ real matrix.

$$B\begin{pmatrix} i_1 & \cdots & i_p \\ j_1 & \cdots & j_p \end{pmatrix}$$

denotes the determinant obtained from B by deleting all rows and columns apart from those of indices $i_1, \ldots, i_p$ and $j_1, \ldots, j_p$ respectively.

A matrix A is said to be Totally Positive—T.P. (strictly—S.T.P.)—if every minor is nonnegative (positive). A square matrix A is oscillating when A is T.P. and some iterate A^m is S.T.P. An oscillating matrix is of weak type r ($r = 1, 2, \ldots, n-2$) if A is T.P., nonsingular, and also satisfies

$$\begin{aligned}
&A\begin{pmatrix} 1, 2, \ldots, r \\ n-r+1, \ldots, n \end{pmatrix} > 0, \\
&A\begin{pmatrix} 1, 2, \ldots, r, r+i \\ i-1, n-r+1, \ldots, n \end{pmatrix} > 0, \qquad i = 2, \ldots, n-r, \\
&A\begin{pmatrix} 1, 2, \ldots, r, r+i \\ i+1, n-r+1, \ldots, n \end{pmatrix} > 0, \qquad i = 1, 2, \ldots, n-r-1.
\end{aligned} \tag{1.2}$$

THEOREM 1.1. *Let $A = \|a_{ij}\|_{i,j=1}^{n}$ be a strictly totally positive (S.T.P.) matrix or more generally an oscillating matrix of weak type r. Consider the polynomial of degree $n - r$*

$$\Delta_r(\lambda) = \det\|a_{ij} - \lambda\delta_{i-r,j}\|_{i,j=1}^{n}, \tag{1.3}$$

where $\delta_{k,l}$ is the usual kronecker delta function. Then $\Delta_r(\lambda)$ possesses exactly $n - r$ simple real zeros of sign $(-1)^r$.

The example of (1.1) is subsumed since

$$A = \left\|\binom{j}{i}\right\|$$

is indeed an oscillating matrix of weak type r (verified in Corollary 2.1 of Sec. 2).

The scope and wide manifestations of totally positive matrices and kernels occur in such diverse contexts as probability theory (e.g., [6], [8]), approximation theory (e.g., [9], [15]), for facilitating numerical procedures in solving certain types of differential equations (e.g., [12]), in the analysis of certain integral and differential operators (e.g., [1], [2], [6], [11], [13]), relevant for the theory of inequalities (e.g., [9], [16]), and elsewhere.

The proof of Theorem 1.1 is quite simple proceeding by an application of the Sylvester determinant identity (see Sec. 2) which reduces considerations to a problem of locating the eigenvalues of an associated oscillating matrix $B = \|B_{ij}\|$ of order $n - r$. In fact, we will obtain the identity

$$\Delta_r(\lambda) = c \det\|B - (-1)^r \lambda b I\|, \tag{1.4}$$

with $c \neq 0$, $b > 0$ (see Sec. 2 for details). Appeal is then made to the important result of Gantmacher and Krein to the effect that an oscillating matrix admits only simple positive eigenvalues. Lipow and Schoenberg [10] in dealing with (1.1) also reduce the analysis to ascertaining the eigenvalues of an oscillating matrix. The reduction process they employ seems more elaborate and special.

Another context in which polynomials of the kind (1.1) occur pertains to properties of eigenfunctions for integral operators induced by Totally Positive kernels (see [2], [4], [5] for further background). We describe the set-up in the matrix case. We will find that the conclusion of Theorem 1.1 falls out as a small part of a rich oscillating structure endowed to certain determinantal polynomial systems associated with oscillation matrices. Let $A = \|a_{ij}\|$ be an $n \times n$ S.T.P. matrix. Designate by

$$u_i(\lambda), \qquad i = 1, 2, \ldots, n \tag{1.5}$$

the algebraic cofactors respectively of the last row in $(A - \lambda I)$. In particular

$$u_1(\lambda) = (-1)^{n+1} \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{22} - \lambda & a_{23} & \cdots & a_{2,n-1} & a_{2,n} \\ a_{32} & a_{33} - \lambda & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} - \lambda & a_{n-1,n} \end{vmatrix}, \tag{1.6}$$

and note the resemblance to determinants of the type (1.3) with $r = 1$.

Clearly,

$$\sum_{j=1}^{n} a_{ij}u_j(\lambda) = \lambda u_i(\lambda), \qquad i = 1, 2, \ldots, n-1,$$

$$\sum_{j=1}^{n} a_{nj}u_j(\lambda) = \lambda u_n(\lambda) + D_n(\lambda) \tag{1.7}$$

where $D_n(\lambda) = \det[A - \lambda I]$. Direct inspection, where A is S.T.P., reveals that $u_i(\lambda)$, $i = 1, 2, \ldots, n-1$ are polynomials of exact degree $n-2$ while $u_n(\lambda) = D_{n-1}(\lambda)$ is of degree $n-1$.

If $D(x, y, \lambda)$ denotes the numerator of the resolvant function associated with an integral operator for the kernel $K(x, y)$, defined on $[a, b]$, then the system $D(k, \lambda) = u_k(\lambda)$ can be interpreted as the discrete version of $D(x, y, \lambda)$ with specification $y = b$ so that $D(k, \lambda)$ corresponds to $D(x, b, \lambda)$. The importance and relevance of $D(x, b, \lambda)$ and generally $D(x, y, \lambda)$ in analyzing the nature of the integral operator $A\phi = \int K(x, y) \phi(y)\, d\mu(y)$, especially where $K(x, y)$ is S.T.P. or oscillating is well established (e.g., see Gantmacher and Krein [2]).

The significance of the system $\{u_i(\lambda)\}$ for the study of the eigenstructure of a S.T.P. matrix A is partially described in [5]. The main facts for $\{u_i(\lambda)\}$ to be proved in this manuscript are now stated (see [7] for other remarkable properties of the system $\{u_i(\lambda)\}$). The results undoubtedly bear analogs in the integral operator case.

THEOREM 1.2. *Let A be S.T.P. of order n. Define $\{u_i(\lambda)\}_1^n$ as in* (1.5). *Then*

(a) *$u_i(\lambda)$ possesses $i-1$ positive and $n-i-1$ (when $i = n$, interpret $n - i - 1 = 0$) negative simple zeros, $i = 1, 2, \ldots, n$;*

(b) *the zeros of $u_i(\lambda)$ and $u_{i+1}(\lambda)$, $i = 1, 2, \ldots, n-1$ strictly interlace.*

A generalized version of Theorem 1.2 involves the cofactors $u_j^{(r)}(\lambda)$, ($j = 1, 2, \ldots, n$) of the rth row of $A - \lambda I_n^{(r)}$ (see Theorem 1.3 for the definition of $I_n^{(r)}$). We establish in Theorem 4.4 of Sec. 4 that $u_r^{(r)}(\lambda)$ has $n-1$ simple zeros; $u_j^{(r)}(\lambda)$, $j \neq r$, has $n-2$ simple zeros. Moreover, $u_j^{(r)}(\lambda)$ and $u_{j+1}^{(r)}(\lambda)$ exhibit strictly interlacing zeros, and we also locate the zeros of $u_j^{(r)}(\lambda)$.

The facts of (a) and (b) bear interpretations in the study of vibrating coupled mechanical systems (e.g., see [2] and [3]). Theorem 1.2 relates to the following generalized characteristic polynomial theorem now highlighted.

THEOREM 1.3. *Let* $A = ||a_{ij}||_1^n$ *be an* $n \times n$ *oscillating matrix. Denote by* $I_n^{(k)} = ||e_{ij}||_{i,j=1}^n$, $k = 0, 1, \ldots, n$ *the diagonal matrix with*

$$e_{ij} = \begin{cases} 0, & i \neq j, \\ -1, & 1 \leqslant i = j \leqslant k, \\ +1, & k+1 \leqslant i = j \leqslant n, \end{cases}$$

so that $I_n^{(0)} = I$ *the identity, and* $I_n^{(n)} = -I$. *Then*

(i) $P_{n,k}(\lambda) = \det||A - \lambda I_n^{(k)}||$ *has* k *negative and* $n-k$ *positive simple zeros.*

(ii) *The zeros of* $P_{n,k}(\lambda)$ *and* $Q_{n,k}(\lambda)$ (*the subdeterminant of* $A - \lambda I_n^{(k)}$ *with* kth *row and column deleted*) *strictly interlace viewed from the origin. This means that separately on the positive and negative axes the zeros of* $P_{n,k}(\lambda)$ *and* $Q_{n,k}(\lambda)$ *strictly interlace displaying the orientation that both on the positive and negative axes* $P_{n,k}(\lambda)$ *exhibits the closer zero to the origin.*

(iii) *The zeros of* $P_{n,k}(\lambda)$ *and* $P_{n,k+1}(\lambda)$ *globally strictly interlace.*

By a standard device of approximating T.P. matrices by S.T.P. matrices [6, p. 88], we obtain the following corollary.

COROLLARY 1.1. *Let* A *and* $I_n^{(k)}$ *be as above, where* A *is T.P.* (*not necessarily oscillatory*). *Then*

(i) $P_{n,k}(\lambda) = \det||A - \lambda I_n^{(k)}||$ *has* n *real zeros, counting multiplicity, of which at most* k *are negative, and at most* $n-k$ *are positive.*

(ii) *The zeros of* $P_{n,k}(\lambda)$ *and* $Q_{n,k}(\lambda)$ *interlace* (*not necessarily strictly*), *viewed from the origin as in Theorem* 1.3.

(iii) *The zeros of* $P_{n,k}(\lambda)$ *and* $P_{n,k+1}(\lambda)$ *interlace.*

The choice of $I_n^{(k)}$ involving a consecutive block of -1's followed by $+1$'s (or in the other order) is crucial for the validity of Theorem 1.3. The theorem fails for general oscillating matrices when $I_n^{(k)}$ remains a diagonal matrix but with ± 1 appearing in arbitrary order (i.e., not necessarily comprised of two blocks of constant opposite sign).

In contrast to the preceding remark, provided A is positive definite, the conclusion of Theorem 1.3 prevails with an arbitrary diagonal $I_n^{(k)}$.

THEOREM 1.4. *Let A be a positive definite $n \times n$ matrix. Define $I_i^{(k)} = I^{(k)}_{(i_1,i_2,\ldots,i_k)} = \|e_{ij}\|$ where $e_{ij} = 0$ for $i \neq j$, $e_{i_\nu i_\nu} = -1, \nu = 1,\ldots,k$ and $e_{jj} = +1, j \neq i_1, i_2,\ldots, i_k$. Then $P_{(i_1,\ldots,i_k)}(\lambda) = P_i(\lambda) = \det[A - \lambda I_i^{(k)}]$ has k negative and $n - k$ positive zeros (not necessarily simple). Moreover, the zeros of $P_{(i_1,\ldots,i_k)}(\lambda)$ and $P_{(i_1,i_2,\ldots,i_k,i_{k+1})}(\lambda)$ interlace (not necessarily strictly), where i_{k+1} is any new index appended to $(i_1, i_2,\ldots, i_k)$.*

REMARK. The first part of this theorem may be obtained as an easy consequence of the Inertia Theorem for symmetric matrices.

Aside from the intrinsic interest of Theorems 1.1–1.4, as mentioned earlier, applications are forthcoming to the theory of interpolation of data by splines at equidistant points (see Lipow and Schoenberg [10] and Schoenberg and Sharma [14]) and to the oscillation theory of solution sets for certain classes of differential equations.

The organization of the paper runs as follows. Section 2 is devoted to establishing useful notation and recording preliminaries on the Sylvester determinant identity and relevant eigenvalue facts for oscillation matrices. Section 2 also contains a simple transparent proof of Theorem 1.1. The proof of the principal Theorem 1.3 and several ramifications are elaborated in Sec. 3.

The discussion of Theorem 1.2 and extensions are given in Sec. 4. Section 5 is concerned with the positive definite case as enunciated in Theorem 1.4. Refinements, and various counterexamples to some natural conjectures are considered in the concluding section. The appropriate version of Theorems 1.2 and 4.3 expressed for the resolvent kernel $D(x, y, \lambda)$ of an integral operator with T.P. kernel $K(x, y)$ will be expounded elsewhere.

2. PRELIMINARIES AND THE PROOF OF THEOREM 1.1

We shall exploit substantially Sylvester's determinant identity stated immediately below for ready reference (cf. [6, p. 3]).

Let A be a fixed $n \times n$ matrix. Specify two sets of p tuples of indices $1 \leqslant \nu_1 < \cdots < \nu_p \leqslant n$, and $1 \leqslant \mu_1 < \cdots < \mu_p \leqslant n$ to be held fixed. For each index i $(1 \leqslant i \leqslant n)$ not contained in the set $\nu = (\nu_1,\ldots, \nu_p)$, and index j $(1 \leqslant j \leqslant n)$ not contained in the set $\mu = (\mu_1,\ldots, \mu_p)$, we form

$$b_{ij} = A\begin{pmatrix} k_1,\ldots, k_{p+1} \\ l_1,\ldots, l_{p+1} \end{pmatrix} \quad \text{[consult (1.1) concerning this notation]}, \tag{2.1}$$

where $(k_1, \ldots, k_{p+1})$ embodies the set of indices $(i, \nu_1, \ldots, \nu_p)$ arranged in natural (i.e. increasing) order and $(l_1, \ldots, l_{p+1})$ embodies the set of indices $(j, \mu_1, \ldots, \mu_p)$ also arranged in natural order. For any selections of indices $i_1 < \cdots < i_q$, $i_m \notin \nu$, and $j_1 < \cdots < j_q$, $j_m \notin \mu$, $q \leqslant n - p$, we have the identity (known as the *Sylvester determinant identity*).

$$B\begin{pmatrix} i_1, \ldots, i_q \\ j_1, \ldots, j_q \end{pmatrix} = \left[A\begin{pmatrix} \nu_1, \ldots, \nu_p \\ \mu_1, \ldots, \mu_p \end{pmatrix}\right]^{q-1} A\begin{pmatrix} \alpha_1, \ldots, \alpha_{q+p} \\ \beta_1, \ldots, \beta_{q+p} \end{pmatrix}, \tag{2.2}$$

where

$$(\alpha_1, \ldots, \alpha_{q+p}) = (i_1, \ldots, i_q, \nu_1, \ldots, \nu_p),$$

$$(\beta_1, \ldots, \beta_{q+p}) = (j_1, \ldots, j_q, \mu_1, \ldots, \mu_p)$$

are each prescribed in natural order.

The submatrix of A, composed of the rows of indices $\nu_1, \ldots, \nu_p$ and columns of indices $\mu_1, \ldots, \mu_p$ common to the determinant (2.1) is called the *pivot block* in the application of Sylvester's determinant identity.

The reader is referred to the statement occurring just prior to Theorem 1.1 in Sec. 1 for the definitions of total positivity (T.P.), strict total positivity (S.T.P.) and oscillating matrices. Some important characterizations and eigenvalue properties of oscillating matrices will now be cited.

THEOREM 2.1 (Gantmacher and Krein [2]; see also [6, Chap. 2]).
(i) *If* $A = \|a_{ij}\|_{i,j=1}^n$ *is T.P., then A is an oscillating matrix iff*

$$|A| \neq 0, \quad a_{i,i+1} > 0, \quad a_{i+1,i} > 0, \quad i = 1, \ldots, n-1. \tag{2.3}$$

(ii) *If A is an oscillating matrix, then any principal submatrix of A is also an oscillating matrix.*

(iii) *If A is an oscillating matrix, then the zeros of* $\det[A - \lambda I] = 0$ *are positive and simple and strictly interlace those of the two* $n-1$ *order principal minors of* $A - \lambda I$*, obtained by deleting the last row and column or the first row and column.*

(iv) *If A is an oscillating matrix, and* $\{u_i(\lambda)\}_{i=1}^n$ *are as in* (1.5), *then* $(-1)^{n-1}u_1(\lambda) > 0$ *for all* $\lambda > 0$ *and* $u_n(\lambda) > 0$ *for all* $\lambda < 0$.

Proof. We validate only the assertion for $u_1(\lambda)$ in (iv). Since A is oscillatory, the zeros of $|A - \lambda I_n|$ strictly interlace the positive zeros of $u_n(\lambda)$ by (iii). Take $\lambda_0 > 0$ such that $\det[A - \lambda_0 I] = 0$ but $u_n(\lambda_0) \neq 0$.

Clearly, $A\mathbf{u}(\lambda_0) = \lambda_0\mathbf{u}(\lambda_0)$, where $\mathbf{u}(\lambda_0)$ denotes the vector with components $[u_1(\lambda_0), \ldots, u_n(\lambda_0)] \not\equiv \mathbf{0}$, and $A^k\mathbf{u}(\lambda_0) = \lambda_0{}^k\mathbf{u}(\lambda_0)$ for $k = 1, 2, \ldots$. For some k, A^k is S.T.P., and from [5], we know that $u_1(\lambda_0) \neq 0$.

The polynomial expansion of $u_1(\lambda)$ is of the form

$$(-1)^{n-1}u_1(\lambda) = \lambda^{n-2}a_{1n} + \lambda^{n-3}\left[\sum_{1<i<n} A\begin{pmatrix}1, i\\ i, n\end{pmatrix}\right] + \cdots + A\begin{pmatrix}1, \ldots, n-1\\ 2, \ldots, \quad n\end{pmatrix},$$

exhibiting only nonnegative coefficients and we have $u_1(\lambda_0) \neq 0$. It follows that $u_1(\lambda) \neq 0$ for all $\lambda > 0$.

Note the following fact.

LEMMA 2.1. *Let A be T.P. (S.T.P.) and let $B = \|b_{ij}\|$ be the $n - p \times n - p$ matrix defined in* (2.1). *Then B is T.P. (S.T.P.).*

Proof. A direct consequence of (2.2).

We are now prepared to prove Theorem 1.1 (see Sec. 1 for its statement).

Proof of Theorem 1.1. Construct the matrix $B = \|b_{ij}\|_{i=r+1,}^{n}\,{}_{j=1}^{n-r}$ with pivot block based on the rows and columns of the minor

$$A\begin{pmatrix}1, 2, \ldots, r\\ n-r+1, \ldots, n\end{pmatrix}$$

via Sylvester's determinant identity. Actually, it is better to apply the Sylvester determinant identity with the same pivot block to the matrix

$$A - \lambda J^{(r)} \tag{2.4}$$

where $J^{(r)} = \|\delta_{i-r,j}\|$ ($\delta_{k,l}$ is the kronecker delta function).

This operation produces a matrix of size $n - r \times n - r$, of the explicit form

$$B - \lambda(-1)^r bI, \qquad b = A\begin{pmatrix}1, 2, \ldots, r\\ n-r+1, n-r+2, \ldots, n\end{pmatrix}. \tag{2.5}$$

By virtue of (2.2), we obtain

$$\det[B - \lambda(-1)^r bI] = b^{n-r-1}\det[A - \lambda J^{(r)}]. \tag{2.6}$$

Because A is *oscillating of weak type* r we may verify directly [invoking repeated appeal to (2.2)] that the conditions (2.3) hold for the matrix B. This means that B is, in fact, oscillating. With this fact in hand Theorem 2.1, part (iii) affirms that the polynomial (2.6) vanishes simply $n - r$ times such that each zero has sign $(-1)^r$. The proof of Theorem 1.1 is complete.

COROLLARY 2.1. *The polynomial* (1.1) *has* $n - r$ *simple zeros of sign* $(-1)^r$.

Proof. It suffices to check that the matrix

$$\left\| \binom{j}{i} \right\|_{i=0,j=k}^{n-k,n}$$

is oscillating of weak type r.

We know that

$$K(i, j) = \binom{j}{i}$$

determines a totally positive kernel as pointed out in [6, p. 139]. The other requirements listed in (1.2) are readily validated with the aid of the following fact:

Given $0 \leqslant i_1 < \cdots < i_s$, $0 \leqslant j_1 < \cdots < j_s$ and $\alpha > -s$, all integers, and let $C = \|C_{kl}\|_{k,l=1}^{s}$,

$$C_{kl} = \binom{\alpha + i_k + j_s}{j_s - j_{s+1-l}}. \tag{2.7}$$

Then

$$\det[C] > 0. \tag{2.8}$$

To prove (2.8) we factor out common terms in each row and column of C, to obtain

$$\det[C] = \frac{\prod_{k=1}^{s} (\alpha + i_k + j_s)!}{\prod_{l=1}^{s} (j_s - j_{s+1-l})!} |E|,$$

where $E = \|d_{kl}\|_{k,l=1}^{s}$, and

$$d_{kl} = \begin{cases} \dfrac{1}{(\alpha + i_k + j_{s+1-l})!} & \text{if} \quad \alpha + i_k + j_{s+1-l} \geqslant 0 \\ 0 & \text{if} \quad \alpha + i_k + j_{s+1-l} < 0. \end{cases}$$

By Lemma 2.2′ [6, p. 107] we have $|C| > 0$ as claimed in (2.8).

With the help of (2.8) the verification of (1.2) for the matrix at hand follows straightforwardly. The conclusion of the corollary is hereby confirmed by virtue of the assertion of Theorem 1.1.

We conclude this section with a final ancillary theorem needed for our analyses in Secs. 3 and 4. First, this definition.

DEFINITION 2.1. Let $x = (x_1, x_2, \ldots, x_n)$ be a real vector of n components.

(i) $S^-(x)$ denotes the number of actual sign changes in the sequence $x_1, x_2, \ldots, x_n$ with zero terms discarded.

(ii) $S^+(x)$ counts the maximum number of sign changes achieved in the sequence $x_1, \ldots, x_n$, where zero terms are assigned values $+1$ or -1 arbitrarily. For example

$$S^-(2, 0, 1, -1, 0, -3) = 1, \qquad S^+(2, 0, 1, -1, 0, -3) = 5.$$

THEOREM 2.2. *Let* $B = \|b_{ij}\|_{i=1,\, j=1}^{n \quad m}$ $(n \geqslant m)$ *be S.T.P.*$_m$. *Let* $x = Bc$ *with* c *a nontrivial real m-vector. Then*

(i) $S^+(x) \leqslant S^-(c)$;

(ii) *If* $S^+(x) = S^-(c)$ *then the sign of the first* (*and last*) *component of* x [*if zero, the sign given in determining* $S^+(x)$], *agrees with the sign of the first* (*and last*) *nonzero component of* c.

(iii) *If* B *is merely T.P. then* $S^-(x) \leqslant S^-(c)$ *and the stipulation* (ii) *reads as* $S^-(x) = S^-(c)$ *entails that the first* (*and last*) *non-zero components of* x *and* c *coincide in sign.*

Proof. The development of (i) can be easily inferred by the methods of [6, p. 223]. The statement of (ii) appears in the discussion of that reference in the slightly weaker form as stated in (iii). We reduce consideration to this case. Let $S^+(x) = S^-(c) = p$. This equation entails the existence of $p+1$ indices $\{x_{i_\nu}\}_{\nu=1}^{p+1}$ satisfying $x_{i_\nu} x_{i_{\nu+1}} \leqslant 0$ and $\{c_{j_\mu}\}_{\mu=1}^{p+1}$ such that $c_{j_\mu} c_{j_{\mu+1}} < 0$. Prescribe ε_{i_ν} sufficiently small obeying

$$p = S^+(x + \varepsilon) = S^-(x + \varepsilon) = S^-(x_{i_1} + \varepsilon_{i_1}, \ldots, x_{i_{p+1}} + \varepsilon_{i_{p+1}}).$$

With ε fixed next determine η to satisfy $\tilde{B}\eta = \varepsilon$ where $\tilde{B}$ is the restriction of B to the rows and columns of indices $\{i_\nu\}$ and $\{j_\mu\}$ respectively. Also, assign $\eta_j = 0$ for $j \neq j_\mu$. Clearly $p = S^-(c + \eta)$. From the construction of ε we may infer $p = S^-(x + \varepsilon) = S^+[B(c + \eta)] \leqslant S^-(c + \eta) = p$ using the result of part (i). Thus $S^-(x + \varepsilon) = S^-(c + \eta)$.

Appeal to (iii) proved in [6, p. 223] and continuity establishes the result of (ii).

3. PROOF OF THEOREM 1.3 AND RAMIFICATIONS

Recall that $I_n^{(k)} = ||\varepsilon_i(n, k)\delta_{ij}||$ with $\varepsilon_i(n, k) = -1$ for $1 \leqslant i \leqslant k$ and $+1$ for $k + 1 \leqslant i \leqslant n$. It is convenient to introduce the following notation.

Let $A = ||a_{ij}||_{i,j=1}^n$ be an $n \times n$ matrix. Set

$$P_{n,k}(\lambda) = \det[A - \lambda I_n^{(k)}] \tag{3.1}$$

and denote by

$$\begin{aligned} Q_{n,k,\hat{l}}(\lambda) &= \det{}_{\hat{l}}[A - \lambda I_n^{(k)}] \\ &= \text{the subdeterminant of } A - \lambda I_n^{(k)} \text{ with the } l\text{th row} \\ &\qquad \text{and column deleted.} \end{aligned} \tag{3.2}$$

We suppress the reference to n where no ambiguity can arise and write, more compactly

$$Q_k(\lambda) = Q_{n,k}(\lambda) = Q_{n,k,\hat{k}}(\lambda), \tag{3.3}$$

and introduce the notation $R_k(\lambda) = R_{n,k}(\lambda) = Q_{n,k,\widehat{k+1}}(\lambda)$. We give symbols to the determinants

$$[A - \lambda I_n^{(k)}]\begin{pmatrix} 1, \ldots, k-1, k, k+2, \ldots, n \\ 1, \ldots, k-1, k+1, k+2, \ldots, n \end{pmatrix} = X_{n,k}(\lambda) = X_k(\lambda), \tag{3.4}$$

$$[A - \lambda I_n^{(k)}]\begin{pmatrix} 1, \ldots, k-1, k+1, k+2, \ldots, n \\ 1, \ldots, k-1, k, k+2, \ldots, n \end{pmatrix} = Y_{n,k}(\lambda) = Y_k(\lambda), \tag{3.5}$$

$$[A - \lambda I_n^{(k)}]\begin{pmatrix} 1, \ldots, k, k+1, k+3, \ldots, n \\ 1, \ldots, k, k+2, k+3, \ldots, n \end{pmatrix} = U_{n,k}(\lambda),$$

$$[A - \lambda I_n^{(k)}]\begin{pmatrix} 1, \ldots, k, k+2, k+3, \ldots, n \\ 1, \ldots, k, k+1, k+3, \ldots, n \end{pmatrix} = V_{n,k}(\lambda),$$

$$[A - \lambda I_n^{(k)}]\begin{pmatrix} 1,\ldots,k-2,k-1,k+1,\ldots,n \\ 1,\ldots,k-2,k,k+1,\ldots,n \end{pmatrix} = W_{n,k}(\lambda),$$

$$[A - \lambda I_n^{(k)}]\begin{pmatrix} 1,\ldots,k-2,k,k+1,\ldots,n \\ 1,\ldots,k-2,k-1,k+1,\ldots,n \end{pmatrix} = Z_{n,k}(\lambda).$$

Thus (3.4) is the subdeterminant of $A - \lambda I_n^{(k)}$ computed after removing row $k+1$ and column k. Similarly $Y_k(\lambda)$ is the minor of $A - \lambda I_n^{(k)}$ with row k and column $k+1$ deleted, etc.

Finally, designate

$$S_{i,i+1}(\lambda) = S_{n,i,i+1}(\lambda) = [A - \lambda I_n^{(k)}]\begin{pmatrix} 1,2,\ldots,i-1,i+2,\ldots,n \\ 1,2,\ldots,i-1,i+2,\ldots,n \end{pmatrix} \tag{3.6}$$

as the principal minor of $A - \lambda I_n^{(k)}$ with rows and columns of indices i and $i+1$ removed.

We will assume inductively the following properties for $P_{n,k}(\lambda)$, $Q_k(\lambda)$, and $R_k(\lambda)$:

For each oscillating A of order $m \times m$, $m < n$:

(i) $P_{m,k}(\lambda)$ possesses precisely k negative and $m-k$ positive simple zeros.

(ii) The zeros of $Q_{m,k}(\lambda) = Q_{m,k,\hat{k}}(\lambda)$ and $P_{m,k}(\lambda)$ strictly interlace separately on the positive and negative axis (not with respect to the whole axis). $P_{m,k}(\lambda)$, $(k \leqslant m-1)$, necessarily admits a simple zero between 0 and the smallest positive zero of $Q_{m,k}(\lambda)$, and, for $1 \leqslant k$, $P_{m,k}(\lambda)$ exhibits a zero between 0 and the least negative zero of $Q_{m,k}(\lambda)$.

(iii) The zeros of $R_{m,k}(\lambda)$ and $P_{m,k}(\lambda)$ strictly interlace in the same sense as in (ii).

Our immediate objective is to advance the induction to $m = n$, $1 \leqslant k \leqslant n-1$, maintaining (i)–(iii), Lemma 3.3 is a key step in this process.

Note the following preliminary facts partly extending the result of Theorem 2.1, part (iv).

LEMMA 3.1. *Let A be oscillating. The determinant*

$$v_1(\lambda) = [A - \lambda I_n^{(k)}]\begin{pmatrix} 1,2,\ldots,k,k+2,\ldots,n \\ 2,3,\ldots,n \end{pmatrix} \tag{3.7}$$

never vanishes for λ negative. When A is S.T.P., $v_1(\lambda)$ displays precisely $n-2$ positive roots.

Proof. Expanding $v_1(\lambda)$ with due cognizance to the multilinear nature of the determinant, we find the representation

$$v_1(\lambda) = (-\lambda)^{n-2}a_{1,k+1} + (-\lambda)^{n-3}\gamma_2 + (-\lambda)^{n-4}\gamma_3 + \cdots + (-\lambda)\gamma_{n-2} + \gamma_{n-1}, \tag{3.8}$$

where all $\gamma_i \geqslant 0$ $(i = 2,\ldots, n-1)$ since they are sums of minors of A. Note that every contribution coming from the multilinear expansion of the determinant (3.7) is nonnegative for $\lambda < 0$. It suffices to determine one or a group of terms contributing to (3.8) whose sum is strictly positive for negative λ. Consider the cumulative terms arising by taking the $-\lambda$ factor from the last $n-k-1$ rows and columns. The result is

$$(-\lambda)^{n-k-1}z(\lambda) \quad \text{where} \quad z(\lambda) = \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1k} & a_{1k+1} \\ a_{22}+\lambda & a_{23} & \cdots & a_{2k} & a_{2,k+1} \\ \vdots & & & & \\ a_{k2} & & \cdots & a_{kk}+\lambda & a_{k,k+1} \end{vmatrix}.$$

Observe that $z(\lambda)$ (apart from a factor) is precisely the analog of $u_1(\lambda)$ [defined in (1.5)] with respect to the matrix $(A + \lambda I)$ contracted to the first $k+1$ rows and columns. Referring to Theorem 2.1, part (iv), we see that $z(\lambda) > 0$ for $\lambda < 0$. The first half of Lemma 3.1 is proved.

In the case where A is S.T.P. we operate on $v_1(\lambda)$ via Sylvester's determinant identity with pivot block the single element $a_{1,k+1}$ reducing (apart from a positive factor) to the form

$$\det||C - \lambda Ic||, \quad \text{where } C \text{ is S.T.P.,} \qquad c = a_{1,k+1}.$$

Theorem 2.1 informs us that the determinant vanishes simply $n-2$ times on the positive axis.

In a symmetrical manner we prove the following lemma.

LEMMA 3.2. *If A is oscillating, then*

$$v_n(\lambda) = [A - \lambda I_n^{(k)}]\begin{pmatrix} 1, 2, \ldots, k, k+2, \ldots, n \\ 1, 2, \ldots, n-1 \end{pmatrix}$$

never vanishes for $\lambda > 0$. Where A is S.T.P., $v_n(\lambda)$ has exactly $n-2$ simple negative zeros.

We now proceed to the two main lemmas.

LEMMA 3.3. *Assume A is an oscillating matrix of order n and stipulate (i)–(iii) to hold for every oscillating matrix of order $m < n$. Then $X_{n,k}(\lambda) \cdot Y_{n,k}(\lambda) > 0$ [see (3.4) and (3.5)] at the negative zeros of $R_{n,k}(\lambda)$, and $U_{n,k}(\lambda) V_{n,k}(\lambda) > 0$ at the positive zeros of $R_{n,k}(\lambda)$.*

Proof. Consider the matrix $A - \lambda I_n^{(k)}$ and define $v_1(\lambda), v_2(\lambda), \ldots, v_n(\lambda)$ as the algebraic cofactors of the $k + 1$st row. (Of course it is understood that $k + 1 \leqslant n$.) We can recognize

$$v_{k+1}(\lambda) = Q_{n,k,\widehat{k+1}}(\lambda) = R_k(\lambda),$$

and

$$- v_k(\lambda) = X_k(\lambda). \tag{3.9}$$

Citing Lemma 3.1, we see that $\mathbf{v}(\lambda) = [v_1(\lambda), \ldots, v_n(\lambda)]$ is a nontrivial real vector for all negative λ. The vector $\mathbf{v}(\lambda)$ by virtue of its definition satisfies the relations [cf. (1.7)]

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} v_j(\lambda) &= - \lambda v_i(\lambda), \qquad i = 1, 2, \ldots, k, \\ \sum_{j=1}^{n} a_{ij} v_j(\lambda) &= \lambda v_i(\lambda), \qquad i = k + 2, \ldots, n, \end{aligned} \tag{3.10}$$

and obviously

$$\sum_{j=1}^{n} a_{k+1,j} v_j(\lambda) = L_{k+1}(\lambda) = \det[A - \lambda J_n^{(k+1)}] \tag{3.11}$$

where

$$J_n^{(k+1)}(\lambda) = \begin{cases} - \delta_{ij}, & i = 1, 2, \ldots, k, \\ \delta_{ij}, & i = k + 2, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

We write (3.10) and (3.11) compactly displaying the image vector as

$$A\mathbf{v}(\lambda) = [- \lambda v_1(\lambda), \ldots, - \lambda v_k(\lambda), L_{k+1}(\lambda), \lambda v_{k+2}(\lambda), \ldots, \lambda v_n(\lambda)]. \tag{3.12}$$

Consider a *negative zero* λ^0 of $v_{k+1}(\lambda)$. Determine

$$r = \max\{l;\, v_l(\lambda^0) \neq 0,\, l \leqslant k\} \tag{3.13}$$

which is well defined by virtue of Lemma 3.1 asserting that $v_1(\lambda^0) \neq 0$. We establish the following three facts:

(a) $v_r(\lambda^0)L(\lambda^0) \geqslant 0$. [We have dropped the subscript and written $L(\lambda) = L_{k+1}(\lambda)$].
(b) $r = k$.
(c) $L(\lambda^0) \neq 0$.

Proof of (a). Assume to the contrary that

$$v_r(\lambda^0)L(\lambda^0) < 0 \tag{3.14}$$

holds and compare $S^-\{A[\mathbf{v}(\lambda^0)]\}$ and $S^-[\mathbf{v}(\lambda^0)]$. Inspection of (3.12) taking account of (3.14), reveals that necessarily

$$S^-[\mathbf{v}(\lambda^0)] < S^-\{A[\mathbf{v}(\lambda^0)]\}, \quad \text{since} \quad \lambda^0 < 0.$$

This produces an inconsistency with the assertion of Theorem 2.2, part (iii). To avert this difficulty, the validity of (a) is required.

Proof of (b). Suppose $r \leqslant k - 1$ (we pointed out earlier that $r \geqslant 1$). Form the perturbed vector $\mathbf{v}(\lambda^0; \boldsymbol{\epsilon}) = \mathbf{v}(\lambda^0) + \boldsymbol{\epsilon}$ where $\boldsymbol{\epsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ has $\varepsilon_i = 0$, $i \neq r$ and $\varepsilon_r = -\varepsilon \operatorname{sign} v_r(\lambda^0)$, $\varepsilon > 0$.

In effect, we are slightly reducing in magnitude exclusively the component $v_r(\lambda^0)$ and all other components of $\mathbf{v}(\lambda^0)$ are unaltered. Manifestly, for ε positive and sufficiently small

$$S^-\{A[\mathbf{v}(\lambda^0; \boldsymbol{\epsilon})]\} \geqslant S^-\{A[\mathbf{v}(\lambda^0)]\},$$

and observe that the image vector $A[\mathbf{v}(\lambda^0; \boldsymbol{\epsilon})]$ has the $r + 1$th coordinate equal to $-\varepsilon a_{r+1,r} \operatorname{sign} v_r(\lambda^0)$ of opposite sign to $-\lambda^0 v_r(\lambda^0)$ since $-\lambda^0 > 0$ and $a_{r+1,r} > 0$ [see Theorem 2.1, part (i)]. Equivalently we have achieved

$$\{A[\mathbf{v}(\lambda^0; \boldsymbol{\epsilon})]\}_r \{A[\mathbf{v}(\lambda^0; \boldsymbol{\epsilon})]\}_{r+1} < 0. \tag{3.15}$$

A contradiction ensues from (3.15) paraphrasing the proof of part (a). Indeed, if $L(\lambda^0) \neq 0$, then we violate (a) directly, and when $L(\lambda^0) = 0$, then by counting sign changes, using (3.15), it follows that $S^-\{A[\mathbf{v}(\lambda^0; \boldsymbol{\epsilon})]\} = S^-[\mathbf{v}(\lambda^0; \boldsymbol{\epsilon})]$ and that there exists an $s \geqslant k + 2$ for which $v_s(\lambda^0, \boldsymbol{\epsilon}) \neq 0$ or Theorem 2.2, part (i) is contradicted. But this in turn violates part (iii) of Theorem 2.2.

Proof of (c). Assuming $L(\lambda^0) = 0$, we execute the perturbation on the kth component $v_k(\lambda^0)$ done above and are led to the same contradiction as in (a).

The result of (a), (b), and (c) in conjunction establish the inequality

$$v_k(\lambda^0)L(\lambda^0) > 0. \tag{3.16}$$

Introduce next the algebraic cofactors

$$\mathbf{w}(\lambda^0) = [w_1(\lambda^0), \ldots, w_n(\lambda^0)]$$

of the $k + 1$st column of $[A - \lambda I_n^{(k)}]$. Comparisons reveal that

$$w_k(\lambda) = -Y_k(\lambda) \quad \text{and} \quad w_{k+1}(\lambda) = R_k(\lambda) = v_{k+1}(\lambda).$$

The parallel analysis as for the $\mathbf{v}(\lambda)$ system yields

$$-Y_k(\lambda^0)L(\lambda^0) = w_k(\lambda^0)L(\lambda^0) > 0. \tag{3.17}$$

Comparing (3.9), (3.16), and (3.17), we obtain

$$X_k(\lambda^0)Y_k(\lambda^0) > 0, \tag{3.18}$$

at the negative zeros of $R_k(\lambda)$.

To deal with the positive zeros of $R_k(\lambda)$ we proceed completely analogous with all considerations implemented, viewed in opposite order, where, for example, the nth component $v_n(\lambda)$ plays the role of $v_1(\lambda)$, etc. In this way we achieve that at the positive zeros of $R_k(\lambda)$, $U_{n,k}(\lambda)V_{n,k}(\lambda) > 0$.

LEMMA 3.3′. *Under the conditions of Lemma* 3.3, $X_k(\lambda)Y_k(\lambda) > 0$ *at the positive zeros of* $Q_{n,k}(\lambda)$, *and* $W_{n,k}(\lambda)Z_{n,k}(\lambda) > 0$ *at the negative zeros of* $Q_{n,k}(\lambda)$.

The proof works in the same manner as Lemma 3.3, where we note that if $Q_{n,k}(\lambda)$ has negative zeros, then $k - 1 \geqslant 1$.

LEMMA 3.4. *Assume A is oscillating and* (i)–(iii) *hold for* $m < n$. *Then* (i)–(iii) *persist for* $m = n$.

Proof. We apply Sylvester's determinant identity to $A - \lambda I_n^{(k)}$ with pivot block the matrix of the determinant $S_{k,k+1}(\lambda)$ producing the identity

$$P_{n,k}(\lambda)S_{n,k,k+1}(\lambda) = \begin{vmatrix} R_k(\lambda) & X_k(\lambda) \\ Y_k(\lambda) & Q_k(\lambda) \end{vmatrix}. \tag{3.19}$$

We know by virtue of the induction hypothesis [condition (iii) with $m = n - 1$] that

$$S_{n,k,k+1}(\lambda) \text{ changes sign separately at the sets of positive and negative zeros of } R_k(\lambda). \tag{3.20}$$

Moreover, Lemma 3.3 tells us that at the negative zeros of $R_k(\lambda)$, $X_k(\lambda)Y_k(\lambda) > 0$. It follows, referring to the identity (3.19) that

$$P_{n,k}(\lambda)S_{n,k,k+1}(\lambda) < 0 \text{ prevails at the negative zeros of } R_k(\lambda). \tag{3.21}$$

The assertions (3.20) and (3.21) together lead to the inference that $P_{n,k}(\lambda)$ strictly changes sign while traversing the set of negative zeros of $R_k(\lambda)$. Moreover the induction hypothesis (iii) and (3.21) also tells us that $P_{n,k}(\lambda)$ is negative at the least negative zero of $R_k(\lambda)$. These facts coupled with the observations that the $P_{n,k}(0) > 0$ and that the leading coefficient of $P_{n,k}(\lambda)$ has sign $(-1)^{n-k}$ clearly implies properties (i) and (iii) of the next induction step on the negative axis. To achieve the result on the positive axis, we apply the above analysis with $S_{n,k+1,k+2}(\lambda)$ in place of $S_{n,k,k+1}(\lambda)$.

The conclusion of (ii) is secured exactly as above examining the equation (3.19) at the positive zeros of $Q_k(\lambda)$ with reliance on Lemma 3.3′ and then replacing $S_{n,k,k+1}(\lambda)$ by $S_{n,k-1,k}(\lambda)$ to obtain (ii) for the negative zeros of $Q_k(\lambda)$. Lemma 3.4 is fully proved once properties (i)–(iii) are confirmed in the cases of $n = 1, 2$ which are direct.

COROLLARY 3.1. *At all the zeros of $R_k(\lambda)$ we have*

$$v_k(\lambda)v_{k+2}(\lambda) < 0, \qquad k = 1, \ldots, n-2. \tag{3.22}$$

Proof. By Lemma 3.3, we have that at the zeros of $R_k(\lambda)$,

$$0 \neq P_{n,k}(\lambda)S_{k,k+1}(\lambda) = -X_k(\lambda)Y_k(\lambda).$$

So $X_k(\lambda) = -v_k(\lambda)$ never vanishes at the zeros of $R_k(\lambda)$. In a symmetrical way we deduce that $v_{k+2}(\lambda) \neq 0$ at these points. Finally, for λ^0 satisfying $R_k(\lambda^0) = 0$, the vector equation

$$A[\mathbf{v}(\lambda^0)] = [-\lambda^0 v_1(\lambda^0), \ldots, -\lambda^0 v_k(\lambda^0), L_{k+1}(\lambda^0), \lambda^0 v_{k+2}(\lambda^0), \ldots, \lambda^0 v_n(\lambda^0)]$$

[see (3.12)], requires

$$S^-\{A[\mathbf{v}(\lambda^0)]\} \leqslant S^-[\mathbf{v}(\lambda^0)].$$

This is only possible if $v_k(\lambda^0)v_{k+2}(\lambda^0) < 0$.

We can now state Theorem 3.1.

THEOREM 3.1. *Let A be oscillating. Then*

(i) $P_{n,k}(\lambda) = \det[A - \lambda I_n^{(k)}]$ *possesses $n - k$ positive and k negative simple zeros.*

(ii) *The sets of positive and sets of negative zeros of $Q_{n,k}(\lambda)$ and $P_{n,k}(\lambda)$ strictly interlace where the smallest positive and least negative zeros of $P_{n,k}(\lambda)$ are closer to the origin than the corresponding zeros for $Q_{n,k}(\lambda)$.*

(iii) *The zeros of $R_{n,k}(\lambda)$ and $P_{n,k}(\lambda)$ strictly interlace in the same sense as in* (ii).

(iv) *The zeros of $P_{n,k}(\lambda)$ and $P_{n,k+1}(\lambda)$ strictly interlace such that the largest zero of $P_{n,k}(\lambda)$ exceeds that of $P_{n,k+1}(\lambda)$.*

Proof. The proof of (i)–(iii) was accomplished through Lemmas 3.1–3.4. The proof of (iv) is obtained with the aid of the following lemma.

LEMMA 3.5. *Let $p_n(\lambda)$ and $q_n(\lambda)$ be two real polynomials of degree n not identically equal, for which*

(a) *$p_n(\lambda)$ and $q_n(\lambda)$ have n real zeros.*

(b) $(-1)^i p_n(\lambda_i) = (-1)^i q_n(\lambda_i) > 0, i = 1, \ldots, n,$ *where* $\lambda_1 < \cdots < \lambda_n$.

Then the roots of $p_n(\lambda)$ and $q_n(\lambda)$ strictly interlace.

Proof. The result follows immediately from the observation that both $p_n(\lambda)$ and $q_n(\lambda)$ have a zero in $(\lambda_i, \lambda_{i+1})$, $i = 1, \ldots, n-1$, and that $p_n(\lambda) - q_n(\lambda) = 0$ necessarily only at $\lambda = \lambda_1, \ldots, \lambda_n$, each of which is simple.

Completion of the proof of Theorem 3.1. Since $P_{n,k}(\lambda) = L_{k+1}(\lambda) - \lambda v_{k+1}(\lambda)$, and $P_{n,k+1}(\lambda) = L_{k+1}(\lambda) + \lambda v_{k+1}(\lambda)$, then $P_{n,k}(\lambda) = P_{n,k+1}(\lambda)$ at the $n - 1$ simple zeros of $v_{k+1}(\lambda)$, and also at $\lambda = 0$. By parts (ii) and (iii) of Theorem 3.1, it follows that $P_{n,k}(\lambda)$ alternates in sign at the set of points including the zeros of $v_{k+1}(\lambda)$ and 0. Applying Lemma 3.5 and utilizing part (i) of Theorem 3.1, the proof is completed.

The interlacing result enunciated in part (iv) of Theorem 3.1 hints at the monotonicity property stated in the next proposition.

PROPOSITION 3.1. *Let A be as in Theorem* 3.1, *and consider the family of matrices depending on the parameter b introduced into the $k+$ 1st row of A, viz.*

$$A_{b,k+1} = \|a_{ij}(b, k+1)\|_{i,j=1}^{n},$$

where

$$a_{ij}(b, k+1) = \begin{cases} a_{ij}, & i = 1,\ldots,k, k+2,\ldots,n; \quad j = 1,\ldots,n, \\ ba_{k+1,j}, & i = k+1; \quad j = 1,\ldots,n. \end{cases}$$

Define $P_{n,k}(\lambda; b) = \det[A_{b,k+1} - \lambda I_n^{(k)}]$. Then the zeros of $P_{n,k}(\lambda, b)$ strictly increase as b increases from $-\infty$ to ∞.

Proof. Expand $P_{n,k}(\lambda; b)$ by the $k+$ 1st row to obtain

$$P_{n,k}(\lambda; b) = bP_{n,k}(\lambda) + (b-1)\lambda R_{n,k}(\lambda). \tag{3.23}$$

Consider the function $-\lambda R_{n,k}(\lambda)/P_{n,k}(\lambda)$. We know from Theorem 3.1, part (iii), that the zeros of $\lambda R_{n,k}(\lambda)$ strictly interlace the zeros of $P_{n,k}(\lambda)$. A familiar analysis implies the representation

$$\frac{-\lambda R_{n,k}(\lambda)}{P_{n,k}(\lambda)} = \sum_{i=1}^{n} \frac{c_i}{\lambda - \lambda_i} + 1,$$

where $\lambda_n < \cdots < \lambda_1$ are the zeros of $P_{n,k}(\lambda)$ and c_i are of one strict sign (in fact, in the case at hand, we have $c_i > 0$).

It follows that $-\lambda R_{n,k}(\lambda)/P_{n,k}(\lambda)$ is monotone decreasing in each of the intervals $(\lambda_{i+1}, \lambda_i)$, $i = 0, 1,\ldots,n$, where $\lambda_0 = \infty$, $\lambda_{n+1} = -\infty$, and

$$\lim_{\lambda\downarrow\lambda_i} \frac{-\lambda R_{n,k}(\lambda)}{P_{n,k}(\lambda)} = \infty, \qquad i = 1,\ldots,n,$$

$$\lim_{\lambda\uparrow\lambda_i} \frac{-\lambda R_{n,k}(\lambda)}{P_{n,k}(\lambda)} = -\infty, \qquad i = 1,\ldots,n,$$

$$\lim_{\lambda\uparrow\lambda_0} \frac{-\lambda R_{n,k}(\lambda)}{P_{n,k}(\lambda)} = \lim_{\lambda\downarrow\lambda_{n+1}} \frac{-\lambda R_{n,k}(\lambda)}{P_{n,k}(\lambda)} = 1.$$

It is now clear that the equality

$$\frac{-\lambda R_{n,k}(\lambda)}{P_{n,k}(\lambda)} = \frac{b}{b-1} \quad \text{[compare to (3.23)]},$$

holds for n distinct numbers $\{\lambda_i(b)\}_{i=1}^n$ which are manifestly the zeros of $P_{n,k}(\lambda; b)$. Since $b/(b-1)$ is monotone we further find that $\lambda_i(b)$ strictly decreases as b increases. A more precise location of their variation is that for $1 < b < \infty$, $\lambda_i < \lambda_i(b) < \lambda_{i-1}$, $i = 1,\ldots, n$, while for $-\infty < b < 1$, $\lambda_{i+1} < \lambda_i(b) < \lambda_i$, $i = 1,\ldots, n$.

Implementing analyses paraphrasing those used in the proof of Theorem 3.1 produces the following result.

THEOREM 3.2. *Let A be oscillating. The $n-k-1$ positive zeros of*

$$[A - \lambda I_n^{(k)}]\begin{pmatrix}1, 2, \ldots, n-1\\ 1, 2, \ldots, n-1\end{pmatrix}$$

strictly interlace the $n-k$ positive zeros of $P_{n,k}(\lambda)$ for $k = 0, 1, \ldots, n-2$, and the $k-1$ negative zeros of

$$[A - \lambda I_n^{(k)}]\begin{pmatrix}2, \ldots, n\\ 2, \ldots, n\end{pmatrix}$$

strictly interlace the k negative zeros of $P_{n,k}(\lambda)$, $k = 2, 3, \ldots, n$.

4. THEOREM 1.2 AND SOME EXTENSIONS

Consider the matrix $A - \lambda I$ with A S.T.P. Define $u_k(\lambda)$, $k = 1, 2, \ldots, n$ as the algebraic cofactors of the last row in $A - \lambda I$. Explicitly,

$$(-1)^{n+k}u_k(\lambda) =$$

$$\begin{vmatrix}
a_{11}-\lambda & a_{12} & \cdots & a_{1,k-1} & a_{1,k+1} & \cdots & a_{1,n-1} & a_{1,n}\\
a_{21} & a_{22}-\lambda & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n-1} & a_{2,n}\\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots\\
a_{k-1,1} & a_{k-1,2} & \cdots & a_{k-1,k-1}-\lambda & a_{k-1,k+1} & \cdots & a_{k-1,n-1} & a_{k-1,n}\\
a_{k,1} & a_{k,2} & \cdots & a_{k,k-1} & a_{k,k+1} & \cdots & a_{k,n-1} & a_{k,n}\\
a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,k-1} & a_{k+1,k+1}-\lambda & \cdots & a_{k+1,n-1} & a_{k+1,n}\\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots\\
a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n-1}-\lambda & a_{n-1,n}
\end{vmatrix}. \tag{4.1}$$

Via Sylvester's Determinant identity with pivot block consisting of the single term $a_{k,n}$ we can convert $u_k(\lambda)$, $(1 \leqslant k \leqslant n-1)$ into the form [cf. (2.6)],

$$(-1)^{n+k} u_k(\lambda) = \frac{1}{[a_{k,n}]^{n-3}} \det \| B + \lambda I_{n-2}^{(k-1)} a_{k,n} \|, \tag{4.2}$$

where

$$B = \| b_{ij} \|_{i,j=1;i,j\neq k}^{n-1},$$

$$b_{ij} = \begin{cases} A\begin{pmatrix} i, & k \\ j, & n \end{pmatrix}, & 1 \leqslant i < k, \\ A\begin{pmatrix} k, & i \\ j, & n \end{pmatrix}, & k < i \leqslant n-1, \end{cases} \qquad (1 \leqslant j \leqslant n-1, j \neq k),$$

and B is S.T.P. in accordance with Lemma 2.1. (Note the difference that λ appears in (4.2) with a plus coefficient.) On the basis of Theorem 3.1, part (i), we may conclude

THEOREM 4.1. *Let A be S.T.P. Then $u_k(\lambda)$, $k = 1, 2, \ldots, n-1$ has exactly $k-1$ simple positive zeros and $n-1-k$ simple negative zeros. $u_n(\lambda)$ has $n-1$ simple positive zeros.*

The next lemma pertains to the interlacing structure of the zeros of the polynomials $u_{n-1}(\lambda)$ and $u_n(\lambda)$.

LEMMA 4.1. *The zeros of $u_{n-1}(\lambda)$ and $u_n(\lambda)$ strictly interlace.*

Proof. Applying Sylvester's determinant identity to $A - \lambda I$ we obtain

$$a(\lambda) D_n(\lambda) = \begin{vmatrix} u_n(\lambda) & -u_{n-1}(\lambda) \\ \tilde{u}_n(\lambda) & \tilde{u}_{n-1}(\lambda) \end{vmatrix}, \tag{4.3}$$

where

$$a(\lambda) = (A - \lambda I)\begin{pmatrix} 2, 3, \ldots, n-1 \\ 1, 2, \ldots, n-2 \end{pmatrix},$$

$$D_n(\lambda) = \det(A - \lambda I),$$

$$\tilde{u}_n(\lambda) = (A - \lambda I)\begin{pmatrix} 2, 3, \ldots, n \\ 1, 2, \ldots, n-1 \end{pmatrix},$$

$$\tilde{u}_{n-1}(\lambda) = (A - \lambda I)\begin{pmatrix} 2, 3, \ldots, n-1, n \\ 1, 2, \ldots, n-2, n \end{pmatrix}. \tag{4.4}$$

The polynomial $\tilde{u}_n(\lambda)$ is the analog of $u_1(\lambda)$ when constructed in terms of the cofactors of the first row of $A - \lambda I$. It follows as in Theorem 2.1, part (iv), that $\tilde{u}_n(\lambda)$ maintains a constant positive sign for $\lambda > 0$. The polynomial $a(\lambda)$ is a corresponding version for a lower order matrix. So again we have

$$a(\lambda) > 0, \quad \text{for} \quad \lambda > 0.$$

It is known [consult Theorem 2.1, part (iii)] that $u_n(\lambda)$ exhibits $n-1$ simple positive zeros which strictly interlace the n simple positive zeros of $D_n(\lambda)$.

Examining the identity (4.3) at the zeros of $u_n(\lambda)$ since $D_n(\lambda)$ strictly alternates in sign at these points while $\tilde{u}_n(\lambda)$ and $a(\lambda)$ maintain a constant sign we infer that $u_{n-1}(\lambda)$ strictly alternates in sign at the zeros of $u_n(\lambda)$. Noting that $u_{n-1}(\lambda)$ is of degree $n-2$, the assertion of Lemma 4.1 follows.

We can now complete the proof of Theorem 1.2.

THEOREM 4.2. *Let A be S.T.P. The zeros of $u_k(\lambda)$ and $u_{k+1}(\lambda)$ strictly interlace, $k = 1, \ldots, n-1$.*

Proof. We have the transformation equations [cf. (1.7)]

$$A\mathbf{u}(\lambda) = [\lambda u_1(\lambda), \lambda u_2(\lambda), \ldots, \lambda u_{n-1}(\lambda), \lambda u_n(\lambda) + D_n(\lambda)], \tag{4.5}$$

where $\mathbf{u}(\lambda) = [u_1(\lambda), \ldots, u_n(\lambda)]$. The vector $\mathbf{u}(\lambda)$ is nontrivial for all real λ [noted in Theorem 2.1, part (iv)]. If $u_i(\lambda^0) = 0$, $i = 2, 3, \ldots, n-1$ then the inequality

$$S^+\{A[\mathbf{u}(\lambda)]\} \leqslant S^-[\mathbf{u}(\lambda)]$$

required in accordance with Theorem 2.2 and owing to the special form of $A\mathbf{u}(\lambda)$ displayed in (4.5) compels the relation

$$u_{i-1}(\lambda^0)u_{i+1}(\lambda^0) < 0 \qquad \text{at any zero } \lambda^0 \text{ of } u_i(\lambda). \tag{4.6}$$

We know by Lemma 4.1 that $u_n(\lambda)$ alternates in sign at the zeros of $u_{n-1}(\lambda)$. It follows in view of (4.6) that $u_{n-2}(\lambda)$ changes sign at the zeros of $u_{n-1}(\lambda)$. Applying the same reasoning to the zeros of $u_{n-2}(\lambda)$ we deduce the interlacing structure of the zeros of $u_{n-3}(\lambda)$ and $u_{n-2}(\lambda)$. Iterating this argument the desired conclusion of the theorem is validated.

A standard approximation procedure leads to the following theorem.

THEOREM 4.3. *Let A be T.P. (not necessarily strict) and form the associated polynomials*

$$u_1(\lambda), u_2(\lambda), \ldots, u_n(\lambda).$$

Then $u_i(\lambda)$ admit only real zeros. (However

$$\deg[u_i(\lambda)] \leqslant n-2, \qquad 1 \leqslant i \leqslant n-1, \qquad \deg[u_n(\lambda)] \leqslant n-1.)$$

The zeros of $u_i(\lambda)$ and $u_{i+1}(\lambda)$ interlace weakly (i.e. coincidences may occur).

The developments of Theorems 4.1–4.3 can be extended to cover the following situation. Define

$$u_j^{(k)}(\lambda), \qquad k, j = 1, 2, \ldots, n, \tag{4.7}$$

as the algebraic cofactors of the kth row of $[A - \lambda I_n^{(k)}]$.

THEOREM 4.4. *Let A be S.T.P. Then*

(a) *$u_k^{(k)}(\lambda)$ has $n-1$ simple zeros; $u_j^{(k)}(\lambda)$, $j \neq k$, has $n-2$ simple zeros.*

(b) (i) *For $j < k$, $u_j^{(k)}(\lambda)$ has $j-1$ negative zeros and $n-j-1$ positive zeros.*

(ii) *$u_k^{(k)}(\lambda)$ has $k-1$ negative zeros and $n-k$ positive zeros.*

(iii) *For $j > k$, $u_j^{(k)}(\lambda)$ has $j-2$ negative zeros and $n-j$ positive zeros.*

(c) *The zeros of $u_j^{(k)}(\lambda)$ and $u_{j+1}^{(k)}(\lambda)$ strictly interlace, $j = 1, \ldots, n-1$.*

Proof. For $k = 1, n$, the relevant assertions were validated as part of Theorems 4.1 and 4.2. Consider $k = 2, \ldots, n-1$. For ease of notation, write

$$u_j^{(k)}(\lambda) = y_j(\lambda).$$

The proof of parts (a) and (b) for $y_j(\lambda)$ follows by appeal to Theorem 3.1 and with the aid of the Sylvester's determinant identity involving the pivot block consisting of the single element a_{jk}.

For the proof of part (c), we need the following fact.

LEMMA 4.2. $y_{j-1}(\lambda)y_{j+1}(\lambda) < 0$ *at the zeros of* $y_j(\lambda)$, $j = 2,\ldots, n-1$.

Proof. (a) $j = k$. Since A is S.T.P., then owing to Theorem 2.2 we have $S^+\{A[\mathbf{y}(\lambda)]\} \leqslant S^-(\mathbf{y}(\lambda))$. This relation is sufficient to assure that $y_{k-1}(\lambda)y_{k+1}(\lambda) < 0$ at the zeros of $y_k(\lambda)$.

(b) $j = k-1$. Assume $y_{k-2}(\lambda)y_k(\lambda) \geqslant 0$ at a zero λ_0 of $y_{k-1}(\lambda)$. Hence,

$$\begin{aligned} S^-[y(\lambda_0)] &= S^-[y_1(\lambda_0),\ldots, y_{k-2}(\lambda_0)] \\ &\quad + S^-[y_k(\lambda_0),\ldots, y_n(\lambda_0)], \\ &\leqslant S^+[-\lambda_0 y_1(\lambda_0),\ldots, -\lambda_0 y_{k-2}(\lambda_0)] \\ &\quad + S^+[0, L_k(\lambda_0), \lambda_0 y_{k+1}(\lambda_0),\ldots, \lambda_0 y_n(\lambda_0)], \\ &\leqslant S^+[Ay(\lambda_0)]. \end{aligned}$$

Comparing with Theorem 2.2, we conclude that $S^-[\mathbf{y}(\lambda_0)] = S^+[A\mathbf{y}(\lambda_0)]$. A contradiction now emerges in view of part (ii) of Theorem 2.2.

(c) $j < k-1$. Assume $y_{j-1}(\lambda)y_{j+1}(\lambda) \geqslant 0$ at a zero λ_0 of $y_j(\lambda)$. This immediately implies

$$S^-[y(\lambda_0)] \leqslant S^+[Ay(\lambda_0)] - 1, \quad \text{in contradiction to Theorem 2.2.}$$

(d) $j \geqslant k+1$. This is done paralleling the analysis of (b) and (c).

Proof of Theorem 4.4, *part* (c). The analysis of Sec. 3 showed that $y_{k-1}(\lambda)L_k(\lambda) > 0$ at the negative zeros of $y_k(\lambda)$, and $y_{k+1}(\lambda)L_k(\lambda) > 0$ at the positive zeros of $y_k(\lambda)$. Thus, by Lemma 4.2, $y_{k-1}(\lambda)L_k(\lambda) < 0$ at the positive zeros of $y_k(\lambda)$. Since $P_{n,k}(\lambda) = L_k(\lambda)$ at the zeros of $y_k(\lambda)$, and $y_{k-1}(0) < 0$, then using part (ii) of Theorem 3.1 and the above facts, our result follows for $j = k-1$.

Since $y_{k-2}(\lambda)y_k(\lambda) < 0$ at the zeros of $y_{k-1}(\lambda)$, and due to the interlacing properties of $y_k(\lambda)$ and $y_{k-1}(\lambda)$ proved above, and the fact that $y_{k-2}(0)$, $y_k(0)$, and $-y_{k-1}(0)$ are positive, it follows that $y_{k-2}(\lambda)$ has a zero between each two zeros of $y_{k-1}(\lambda)$. But since $y_{k-2}(\lambda)$ has $n-k+1$ positive zeros by part (b), $y_{k-2}(\lambda)$ must have a zero exceeding the largest zero of $y_{k-1}(\lambda)$.

Thus, part (c) is proven for $j = k-2$. In the same manner we prove part (c) for $j < k$, and analogously for $j \geqslant k$.

5. POSITIVE DEFINITE CASE

Throughout this section $A = ||a_{ij}||_{i,j=1}^{n}$ represents an $n \times n$ positive definite matrix. Let

$$I_{n,\mathfrak{i}}^{(k)} = I_{n,(i_1,i_2,\ldots,i_k)}^{(k)} = ||e_{ij}||_{i,j=1}^{n}$$

where

$$e_{ij} = \begin{cases} -\delta_{ij}, & i = i_1, i_2, \ldots, i_k, \\ \delta_{ij}, & i \neq i_1, i_2, \ldots, i_k. \end{cases}$$

For any $k = 0, 1, 2, \ldots, n$, we claim the following theorem.

THEOREM 5.1. *Let A and $I_{n,(i_1,\ldots,i_k)}^{(k)}$ be as above. Then*

(a) $\det[A - \lambda I_{n,\mathfrak{i}}^{(k)}]$ *has k negative and $n - k$ positive zeros.*

(b) *The zeros of* $\det[A - \lambda I_{n,(i_1,i_2,\ldots,i_k)}^{(k)}]$ *and* $\det[A - \lambda I_{n,(i_1,i_2,\ldots,i_{k+1})}^{(k+1)}]$ *weakly interlace where i_{k+1} is any new index appended to $(i_1, i_2, \ldots, i_k)$.*

(c) *The zeros of any $n - 1$ order principal minor of $A - \lambda I_{n,\mathfrak{i}}^{(k)}$ "weakly interlace" those of $A - \lambda I_{n,\mathfrak{i}}^{(k)}$ relative to the origin in the analogous sense as in statement* (ii) *of Theorem* 1.3, *except that coincidences and coalescence of the zeros of these determinants may occur.*

REMARK. (a) may be obtained as a result of the Inertia theorem.

REMARK. Because simultaneous permutations of rows and columns preserves positive definiteness, without loss of generality, we may take $(i_1, i_2, \ldots, i_k, i_{k+1}) = (1, 2, \ldots, k, k+1)$. In this event $I_{n,\mathfrak{i}}^{(k)} = I_n{}^{(k)}$ as defined in the introduction.

I. Suppose inductively (a)–(c) is established for all positive definite matrices of order $m < n$. The following elementary perturbation facts will be helpful.

LEMMA 5.1. *Subject to the induction assumption* I *above, we can perturb any $m \times m$, $m < n$, positive definite matrix B to B_δ, δ sufficiently small, where B_δ is positive definite and $B_\delta \to B$ as $\delta \to 0$, such that* $\det[B_\delta - \lambda I_m{}^{(k)}]$, *and*

$$[B_\delta - \lambda I_m{}^{(k)}]\begin{pmatrix} 1, \ldots, i-1, i+1, \ldots, m \\ 1, \ldots, i-1, i+1, \ldots, m \end{pmatrix}$$

each have distinct roots.

Proof. The proof is by induction on m. For $m = 2$, the lemma is directly verified. We assume the lemma holds for any $m - 1 \times m - 1$ positive definite matrix, and hence we assume that B has been perturbed to $B_{\tilde{\delta}}$, $\tilde{\delta}$ sufficiently small, which we rename B, such that B is positive definite, and

$$[B - \lambda I_m^{(k)}]\begin{pmatrix}1,\ldots,i-1,i+1,\ldots,m\\1,\ldots,i-1,i+1,\ldots,m\end{pmatrix}$$

has distinct roots. Let B_δ be identical with B, except for b_{ii} perturbed to $b_{ii} + \delta$, $\delta > 0$. Thus B_δ is positive definite, and

$$\begin{aligned}&\det[B_\delta - \lambda I_m^{(k)}]\\&= \det[B - \lambda I_m^{(k)}] + \delta[B - \lambda I_m^{(k)}]\begin{pmatrix}1,\ldots,i-1,i+1,\ldots,m\\1,\ldots,i-1,i+1,\ldots,m\end{pmatrix}.\end{aligned}$$

The desired result follows easily using the fact that the roots of

$$[B - \lambda I_m^{(k)}]\begin{pmatrix}1,\ldots,i-1,i+1,\ldots,m\\1,\ldots,i-1,i+1,\ldots,m\end{pmatrix}$$

are distinct and weakly interlace the roots of $\det[B - \lambda I_m^{(k)}]$.

LEMMA 5.2. *We suppose the stipulations of Lemma* 5.1 *hold, and assume B is an $m \times m$, $m < n$, positive definite matrix for which* $\det[B - \lambda I_m^{(k)}]$ *and*

$$[B - \lambda I_m^{(k)}]\begin{pmatrix}1,\ldots,i-1,i+1,\ldots,m\\1,\ldots,i-1,i+1,\ldots,m\end{pmatrix}$$

each have distinct roots. Then we can perturb B to B_ε, ε sufficiently small, where B_ε is positive definite, $B_\varepsilon \to B$ as $\varepsilon \to 0$, such that $\det[B_\varepsilon - \lambda I_m^{(k)}]$ *and*

$$(B_\varepsilon - \lambda I_m^{(k)})\begin{pmatrix}1,\ldots,i-1,i+1,\ldots,m\\1,\ldots,i-1,i+1,\ldots,m\end{pmatrix}$$

do not share a common zero.

Proof. The proof is by induction on m. For $m = 2$, the result can be directly established.

Assume that B is such that

$$[B - \lambda I_m^{(k)}]\begin{pmatrix}1,\ldots,i-1,i+1,\ldots,m\\1,\ldots,i-1,i+1,\ldots,m\end{pmatrix}$$

and

$$[B - \lambda I_m^{(k)}]\begin{pmatrix}1,\ldots,i-2,i+1,\ldots,m\\1,\ldots,i-2,i+1,\ldots,m\end{pmatrix}$$

each have distinct zeros and share no common zero. We may assume as well, in accordance with Lemma 5.1, that $\det[B - \lambda I_m^{(k)}]$ has distinct zeros.

Let B_ε be identical with B except that $b_{i,i+1}$ and $b_{i+1,i}$ are replaced by $b_{i,i+1} + \varepsilon$ and $b_{i+1,i} + \varepsilon$, respectively. For ε sufficiently small, B_ε persists to be positive definite. Observe that

$$\begin{aligned}&\det[B_\varepsilon - \lambda I_m^{(k)}]\\&= \det[B - \lambda I_m^{(k)}] - \varepsilon^2[B - \lambda I_m^{(k)}]\begin{pmatrix}1,\ldots,i-2,i+1,\ldots,m\\1,\ldots,i-2,i+1,\ldots,m\end{pmatrix}\\&\quad - 2\varepsilon[B - \lambda I_m^{(k)}]\begin{pmatrix}1,\ldots,i-2,i,i+1,\ldots,m\\1,\ldots,i-2,i-1,i+1,\ldots,m\end{pmatrix}.\end{aligned} \tag{5.1}$$

The desired conclusion of the lemma can be deduced from (5.1) for ε suitably small as a consequence of the fact that

$$[B - \lambda I_m^{(k)}]\begin{pmatrix}1,\ldots,i-1,i+1,\ldots,m\\1,\ldots,i-1,i+1,\ldots,m\end{pmatrix}$$

and

$$[B - \lambda I_m^{(k)}]\begin{pmatrix}1,\ldots,i-2,i+1,\ldots,m\\1,\ldots,i-2,i+1,\ldots,m\end{pmatrix}$$

share no common zero.

Proof of Theorem 5.1. The cases $n = 1, 2$ are easily proved.

Henceforth we assume the perturbations of Lemmas 5.1–5.2 are effective. Applying Sylvester's determinant identity with pivot block

$$[A - \lambda I_n^{(k)}]\begin{pmatrix}1,\ldots,k-1,k+2,\ldots,n\\1,\ldots,k-1,k+2,\ldots,n\end{pmatrix} = S(\lambda)$$

yields

$$\det[A - \lambda I_n^{(k)}]S(\lambda) = \begin{vmatrix} R_k(\lambda) & C_k(\lambda) \\ D_k(\lambda) & Q_k(\lambda) \end{vmatrix}, \tag{5.2}$$

where

$$R_k(\lambda) = [A - \lambda I_n^{(k)}]\begin{pmatrix} 1, 2,\ldots, k, k+2,\ldots, n \\ 1, 2,\ldots, k, k+2,\ldots, n \end{pmatrix},$$

$$Q_k(\lambda) = [A - \lambda I_n^{(k)}]\begin{pmatrix} 1, 2,\ldots, k-1, k+1,\ldots, n \\ 1, 2,\ldots, k-1, k+1,\ldots, n \end{pmatrix},$$

$$C_k(\lambda) = [A - \lambda I_n^{(k)}]\begin{pmatrix} 1, 2,\ldots, k-1, k, k+2,\ldots, n \\ 1, 2,\ldots, k-1, k+1, k+2,\ldots, n \end{pmatrix},$$

$$D_k(\lambda) = [A - \lambda I_n^{(k)}]\begin{pmatrix} 1,\ldots, k-1, k+1, k+2,\ldots, n \\ 1,\ldots, k-1, k, k+2,\ldots, n \end{pmatrix},$$

$$k = 1, 2,\ldots, n-1.$$

Since A is a symmetric matrix, it is clear that $C_k(\lambda) = D_k(\lambda)$. At the zeros $\lambda_1, \lambda_2,\ldots, \lambda_{n-1}$ of $R_k(\lambda)$ taking account of the nondegeneracy stipulation we have $S(\lambda) \neq 0$.

We see from (5.2) that

$$P_{n,k}(\lambda)S(\lambda) = \det[A - \lambda I_n^{(k)}]S(\lambda) = -[C_k(\lambda)]^2 \leqslant 0,$$

for

$$\lambda = \lambda_1, \lambda_2,\ldots, \lambda_{n-1}. \tag{5.3}$$

By the induction assumption $S(\lambda_i)$ strictly alternates in sign in the manner prescribed in (c), and we therefore deduce

$$P_{n,k}(\lambda_i)P_{n,k}(\lambda_{i+1}) \leqslant 0, \qquad i = 1, 2,\ldots, n-1. \tag{5.4}$$

The conclusions of (a) and (c) with regard to $\det[A - \lambda I_n^{(k)}] = P_{n,k}(\lambda)$ emanates directly based on the information of (5.3) and (5.4). The location of the zeros on the positive or negative axis is readily ascertained where the corresponding facts pertaining to the zeros of $R_k(\lambda)$ are used. The orientation of the zeros of $P_{n,k}(\lambda)$ with respect to the zeros of $R_k(\lambda)$ and $Q_k(\lambda)$ can also be discerned in this manner. (Consult the statement (c) of the theorem.)

The proof of (b) can now be done exploiting all the preceding facts. We omit the remaining formal steps since it involves repetition of the style of reasoning heavily exploited earlier in this paper.

6. EXTENSION AND COUNTEREXAMPLES

It is natural to inquire as to what extent do the theorems of Secs. 2–5 remain in force for matrices which are not oscillatory or positive definite. We find that the results indeed persist for matrices conjugate to an oscillatory matrix subject to certain restrictions. The precise assertions are the content of Theorem 6.1 and its corollaries.

THEOREM 6.1. *Let A be an $n \times n$ oscillatory or positive definite matrix and suppose $U = ||u_{ij}||_{i,j=1}^{n}$ is nonsingular and commutes with $I_n^{(k)}$. Thus $u_{ij} = 0$ for*

$$\begin{Bmatrix} 1 \leqslant i \leqslant k < j \leqslant n \\ 1 \leqslant j \leqslant k < i \leqslant n \end{Bmatrix}.$$

Then $\det[U^{-1}AU - \lambda I_n^{(k)}]$ *vanishes at k negative and $n - k$ positive values, which are distinct in the case where A is oscillatory.*

Proof. The proof ensues instantly by multiplying $|U^{-1}AU - \lambda I_n^{(k)}|$ by $|U|$ on the left and by $|U^{-1}|$ on the right, and observing that the characteristic polynomials $|A - \lambda I_n^{(k)}|$ and $|U^{-1}AU - \lambda I_n^{(k)}|$ share identical roots.

COROLLARY 6.1. *Let A and U be as above. If U commutes with both $I_n^{(k)}$ and $I_n^{(k+1)}$, then the roots of $|U^{-1}AU - \lambda I_n^{(k)}|$ and $|U^{-1}AU - \lambda I_n^{(k+1)}|$ interlace, and the interlacing is strict for A oscillatory.*

Proof. See proof of Theorem 6.1.

REMARK. The property of interlacing of the roots of the principal minors as in Theorem 3.1 and 5.1 does not carry over to the above situation.

COROLLARY 6.2. *If $B = A^{-1}$, where A is an oscillatory matrix, then B satisfies the conditions of Theorem* 6.1 *and Corollary* 6.1.

Proof. Let $K = ||k_{ij}||_{i,j=1}^{n}$ where $k_{ij} = \delta_{ij}(-1)^i$, $i = 1, \ldots, n$. Then $K^{-1} = K$, and $B = K^{-1}CK$ where C is an oscillatory matrix, since

$$B\begin{pmatrix} i_1,\ldots,i_p \\ j_1,\ldots,j_p \end{pmatrix} = (-1)^{\sum_{m=1}^{p}(i_m+j_m)} \frac{A\begin{pmatrix} j_1',\ldots,j_{n-p}' \\ i_1',\ldots,i_{n-p}' \end{pmatrix}}{|A|},$$

where $\{j_1',\ldots,j_{n-p}'\}$ and $\{i_1',\ldots,i_{n-p}'\}$ are complementary sets of indices to $\{j_1,\ldots,j_p\}$ and $\{i_1,\ldots,i_p\}$, respectively in $\{1,\ldots,n\}$ (cf. [6, p. 3]).

PROPOSITION 6.1. *Let A be an $n \times n$ oscillatory or positive definite matrix. Then $A - \lambda I_{n,\mathbf{b}}^{(k)}$ satisfies Theorems* 3.1 *and* 5.1 *respectively, where*

$$I_{n,\mathbf{b}}^{(k)} = \delta_{ij} b_i, \qquad j, i = 1,\ldots,n,$$

and $b_i < 0$, $i = 1,\ldots,k$; $b_i > 0$, $i = k+1,\ldots,n$.

Proof. Divide the ith row and ith column of

$$A - \lambda I_{n,\mathbf{b}}^{(k)} \quad \text{by} \quad \sqrt{|b_i|}, \qquad i = 1,\ldots,n.$$

COROLLARY 6.3. *Let A be as in Proposition* 6.1. *Let V and W be arbitrary $n \times n$ nonsingular positive diagonal matrices. Then $VAW - \lambda I_n^{(k)}$ inherits the properties enunciated in Theorems* 3.1 *and* 5.1.

COUNTEREXAMPLES

(1) It is tempting to conjecture the validity, with natural modifications, of the results of Sec. 3 for strict sign regular matrices. (These are matrices where all the minors of a prescribed order—say pth maintain the same sign ε_p, $\varepsilon_p = +1$ or -1. Strictly totally positive matrices have $\varepsilon_p = +1$.) One could also contemplate perhaps that the result of Theorem 5.1 on positive definite matrices persists for the case where A is merely symmetric. Both these conjectures are quickly settled negatively by the example

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

In fact the roots of

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1+\lambda \end{vmatrix}$$

are complex (actually purely imaginary).

(2) Let A be oscillatory and consider the "characteristic polynomial"

$$\det[A - \lambda I^{(k)}_{n,(i_1,i_2,\ldots,i_k)}], \tag{6.1}$$

(the notation is that of Sec. 5). $I^{(k)}_{n(i_1,i_2,\ldots,i_k)}$ exhibits -1 on the diagonal in the rows of indices $i_1, i_2, \ldots, i_k$ and $+1$ elsewhere on the diagonal.

Some roots of (6.1) may be complex. Take

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix}$$

and $k = 1$, $i_1 = 2$, so

$$\det[A - \lambda I^{(k)}_{3,2}] = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 2 & 2+\lambda & 1 \\ 2 & 2 & 1-\lambda \end{vmatrix}.$$

Its roots are 0, $\sqrt{-1}$ and $-\sqrt{-1}$.

An appropriate small perturbation of A leads to a S.T.P. matrix, but where (6.1) continues to manifest complex roots.

These preceding examples point up the fact that in dealing with oscillatory matrices the division of the $+1$ and -1's on the diagonal of $I_n^{(k)}$ into at most two blocks is crucial. This is in sharp contrast to the case of positive definite matrices.

One of the factors for the difference is this: The eigenvalues of an $(n-1) \times (n-1)$ principal minor of a positive definite matrix interlace the eigenvalues of the matrix while the corresponding property fails to hold for oscillatory matrices.

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